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Probability of Collision of Aircraft with Dissimilar Position Errors

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The natural safety metric in air traffic management is the probability of collision between aircraft; due to its extremely low probability of occurrence, it is virtually impossible to measure directly in simulations. It is shown that the rms position error is an indirect safety metric because it specifies an upper bound for the probability of collision. The latter can be calculated from the rms position error of two aircraft, whether it is the same σ (similar aircraft) or different σ_1 and σ_2 (dissimilar aircraft); the latter more general case is considered here. The rms position error is easy to implement as a safety metric: it just requires monitoring of the deviation between the actual trajectory and the intended trajectory. The actual trajectory could be specified by real flight data or a real-time or fast-time simulation; it could include en route flying, terminal maneuvers, and ground movements, as long as a minimum separation distance is defined for each phase. It is shown that International Civil Aviation Organization (ICAO) target level of safety (TLS) of low probability of collision is attained if the rms position error σ does not exceed about $\frac{1}{11}$th $\sigma \leq 0.09L$ of the minimum separation distance L. This assumes both aircraft have the same rms position error; in this sense they are similar aircraft. In the case of dissimilar aircraft, with distinct rms position errors, $\sigma_1 > \sigma_2$, the probability of collision will increase by up to a factor of five if σ_1 and σ_2 do not differ by more than a factor of nine. This is a relatively small increase, which remains within ICAO target level of safety (TLS) of 5×10^{-9} per flight hour, if $L/\sigma \geq 12$ is raised, that is, the rms error is about $\frac{1}{12}$th of the separation distance.

I. Introduction

THE International Civil Aviation Organization (ICAO) target level of safety (TLS) standard for safety[1–3] specifies a collision probability no greater than 5×10^{-9} per flight hour. This can be obtained from the collision probability per nautical mile, by multiplying by the speed in knots. The probability of collision per flight can be obtained by multiplying the probability of collision per nautical mile (per hour) by the length (duration) of the flight in nautical miles (hours). Of the three probabilities of collision, per hour, per flight, and per nautical mile, the latter will be used because it is equally easy to convert to the other two and better suited to the analysis to be presented. The probability of collision is an exceedingly small number, and, thus, direct measurement, for example, by simulating 10^{11} flight hours, is not practicable. A good indirect measure of an upper bound for the probability of collision[4] is the rms position error: It will be shown that if the rms position error σ is no more than $\frac{1}{11}$th $\sigma \leq L/11$ of the minimum separation distance, then the ICAO TLS will be ensured. This result is obtained using what is arguably the simplest model, representing the aircraft as point masses, with equal rms position errors, moving in unbounded space. The probabilities of collision are treated as one dimensional, on the assumption that along-track, cross-track, and altitude errors are statistically independent. In the present paper one of these restrictions is removed, namely, that the rms position error be the same for both aircraft. The rms position error may be the consequence of atmospheric disturbances[5–8] or flight-path deviations related to airplane stability[9–11] and need not be the same for the two aircraft. The paper is concerned with collision probability, as an indication of the eventual need for collision avoidance maneuvers.[12]

Instead of assuming that both aircraft have the same rms position error σ, in the present work the results apply as well to the case when the two aircraft have different rms position errors σ_1 and σ_2 (see Sec. II). The upper bound for the probability of collision (Sec. II.A) can be used to calculate maximum (Sec. II.B) and cumulative (Sec. II.C) probability of collision. It is found that the collision probabilities for dissimilar aircraft can be partially reduced (Sec. III) to those for similar aircraft: 1) The cumulative probability of collision is the same, replacing the variances of the position errors $(\sigma_1)^2$ and $(\sigma_2)^2$ by their arithmetic mean $2(\bar{\sigma})^2 = (\sigma_1)^2 + (\sigma_2)^2$ (Sec. III.A). 2) The maximum probability of collision involves also the geometric mean of the variances of position error $\sigma_1\sigma_2$ (Sec. III.B). Result 2 differs from the usual bivariate Gaussian probability distribution 1 and leads to an aircraft dissimilarity function $f \equiv \bar{\sigma}^2/\sigma_1\sigma_2$, which depends only on an aircraft dissimilarity factor $\lambda \equiv \sigma_2/\sigma_1$ (Sec. III.C). It can be shown (Sec. IV) that if the rms errors of position do not differ by more than a factor of nine, $\sigma_2 \leq 9\sigma_1$, then the collision probability increases by less than a factor of five, for horizontal separation in controlled (Sec. IV.A) and noncontrolled (Sec. IV.B) airspace or for altitude separation (Sec. IV.C). It is argued in (Sec. V) that this safety metric based on rms position error can be applied to any situation in which a minimum separation distance can be defined, for example, not only en route flying, but also approach to land, climb after takeoff, and airport surface movements.

II. Maximum and Cumulative Probabilities of Collision for Dissimilar Aircraft

An upper bound for the probability of collision of two aircraft is calculated by considering the worst possible collision scenario. The aircraft may have dissimilar rms position errors σ_1 and σ_2, and the calculation (Sec. II.A) includes the maximum (Sec. II.B) and cumulative (Sec. II.C) probabilities of collision.

A. Probability of Collision for Dissimilar Aircraft

To be sure of meeting the ICAO TLS standard, an upper bound for the probability of collision should be calculated; if the upper bound is below the limit, then the probability of collision will also surely be below the limit. An upper bound for the probability of collision can be calculated for the worst possible collision scenario. The probability of collision increases the longer the aircraft are kept at minimum separation distance L. Thus, the worst cases are for two aircraft flying all of the time at the minimum collision distance L, on parallel tracks (Fig. 1). One extreme subcase is (Fig. 2) when the aircraft fly on the same path at a distance L, one behind the other; a collision occurs if they drift in position by x and $L - x$, respectively,

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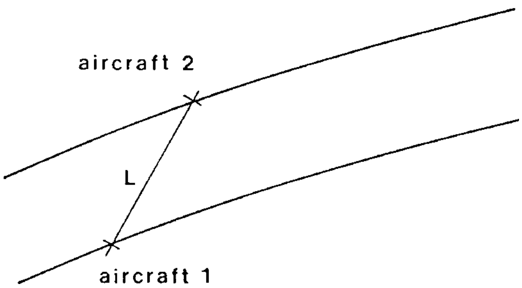

Fig. 1 Two aircraft flying always at minimum separation distance *L*, thus implying parallel paths and equal velocity.

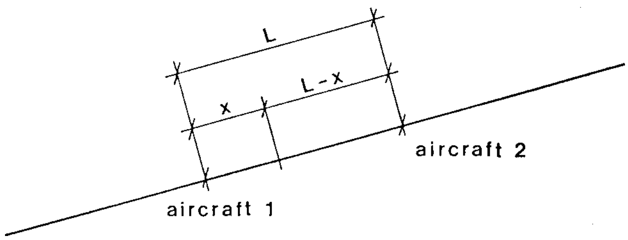

Fig. 2 Subcase of aircraft with the same trajectory and longitudinal separation *L*, which collide if the longitudinal position errors are *x* and *L* − *x* in opposite directions.

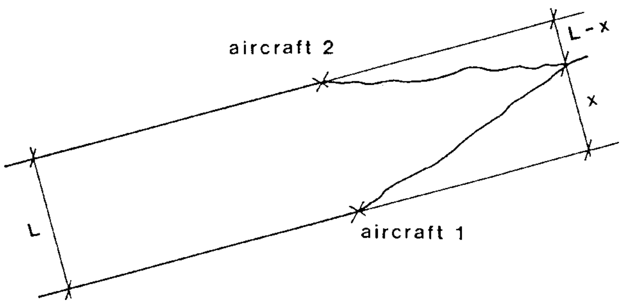

Fig. 3 Subcase of aircraft with parallel trajectories at distance equal to the separation *L*, which collide if the lateral position errors are *x* and *L* − *x* in opposite directions.

one toward the other, for any x. Another extreme subcase (Fig. 3) is two aircraft on parallel paths at a distance L. A collision occurs if one aircraft has a lateral deviation x, and the other has a lateral deviation $L - x$, in opposite directions, for any x. Thus, for all subcases, the probability of collision at position x is

$$P_{12}(x) = P_1(x) P_2(L - x) \tag{1}$$

where $P_1(x)$ is the probability of deviation x for aircraft 1 and $P_2(L - x)$ the probability of deviation $L - x$ for aircraft 2, the two being considered independent.

It can be argued, on the basis of the central limit theorem of the theory of statistics, that the probability distribution should be Gaussian, with zero mean. Thus, the probability of deviation x for the first aircraft is

$$P_1(x) = \left[1\big/\left(\sigma_1\sqrt{2\pi}\right)\right] \exp\left\{-x^2\big/\left[2(\sigma_1)^2\right]\right\} \tag{2}$$

where σ_1 is the rms position error, and similarly for the second aircraft

$$P_2(L - x) = \left[1\big/\left(\sigma_2\sqrt{2\pi}\right)\right] \exp\left\{-(L - x)^2\big/\left[2(\sigma_2)^2\right]\right\} \tag{3}$$

which may have a distinct rms position error σ_2. The collision probability (1) is, thus, given by

$$P_{12}(x) = [1/(2\pi\sigma_1\sigma_2)] \exp\left\{-\left[(x/\sigma_1)^2 + ((L-x)/\sigma_2)^2\right]\big/2\right\} \tag{4}$$

for dissimilar aircraft, $\sigma_1 \neq \sigma_2$. In the case of similar aircraft, in the sense of having the same rms position error, $\sigma_1 = \sigma_2 \equiv \sigma$:

$$P_{12}(x) = [1/(2\pi\sigma^2)] \exp\{-[x^2 + (L - x)^2]/(2\sigma^2)\} \tag{5}$$

The probability of collision simplifies to Eq. (5).

B. Maximum Probability of Collision

The extremum of the probability of collision (4) occurs when its derivative

$$\frac{\mathrm{d}P_{12}(x)}{\mathrm{d}x} = -\left[\frac{1}{\pi\sigma_1\sigma_2}\right] \exp\left\{-\left[\left(\frac{x}{\sigma_1}\right)^2 + \left(\frac{L-x}{\sigma_2}\right)^2\right]\Big/2\right\} \times \left[\frac{x}{(\sigma_1)^2} + \frac{x - L}{(\sigma_2)^2}\right] \tag{6}$$

vanishes,

$$\frac{\mathrm{d}P_{12}(x_m)}{\mathrm{d}x_m} = 0: x_m = \frac{L}{1 + (\sigma_2/\sigma_1)^2} \tag{7}$$

The position of the extremum of the probability of collision or the most likely position for collision

$$\begin{aligned} &\text{case} \quad \sigma_2 < \sigma_1, && \text{position} \quad x_m > L/2 \\ &\text{case} \quad \sigma_2 = \sigma_1, && \text{position} \quad x_m = L/2 \\ &\text{case} \quad \sigma_2 > \sigma_1, && \text{position} \quad x_m < L/2 \end{aligned} \tag{8}$$

is 1) the midline $x_m = L/2$ if the rms position error is the same for both aircraft or 2) if the rms position error is different for the two aircraft, the position for the extremum of probability of collision for collision is shifted toward the aircraft with smallest position error, that is, the aircraft with largest position error drifts more toward the collision. Thus, the most likely position for collision is closest to the intended track of the aircraft with smaller rms position error. This is easily understood by representing the aircraft headon (Fig. 4) each with its own Gaussian probability distribution for the lateral deviation. The most likely position for collision is closest to the aircraft with smaller rms position error.

It has to be checked that the extremum of the probability of collision is a maximum and not a minimum. This needs consideration of the second-order derivative of the probability of collision (4), that is, the next order derivative after Eq. (6):

$$\frac{\mathrm{d}^2 P_{12}(x)}{\mathrm{d}x^2} = -\left[\frac{1}{\pi\sigma_1\sigma_2}\right] \exp\left\{-\left[\left(\frac{x}{\sigma_1}\right)^2 + \left(\frac{x-L}{\sigma_2}\right)^2\right]\Big/2\right\} \times \left\{\frac{1}{(\sigma_1)^2} + \frac{1}{(\sigma_2)^2} - 2\left[\frac{x}{(\sigma_1)^2} + \frac{x-L}{(\sigma_2)^2}\right]^2\right\} \tag{9}$$

which is negative at the position x_m given by Eq. (7), namely,

$$\frac{\mathrm{d}^2 P_{12}(x_m)}{\mathrm{d}^2 x_m} = -\left[\frac{1}{\pi\sigma_1\sigma_2}\right] \cdot \left[\left(\frac{1}{\sigma_1}\right)^2 + \left(\frac{1}{\sigma_2}\right)^2\right] \times \exp\left\{-\left[\left(\frac{x_m}{\sigma_1}\right)^2 + \left(\frac{L - x_m}{\sigma_2}\right)^2\right]\Big/2\right\} < 0 \tag{10}$$

so that the probability of collision is a maximum at $x = x_m$,

$$P_m \equiv P_{12\,\max} = P_{12}(x_m) = [1/(2\pi\sigma_1\sigma_2)] \exp\left\{-(L^2/2)\big/\left[(\sigma_1)^2 + (\sigma_2)^2\right]\right\} \tag{11}$$

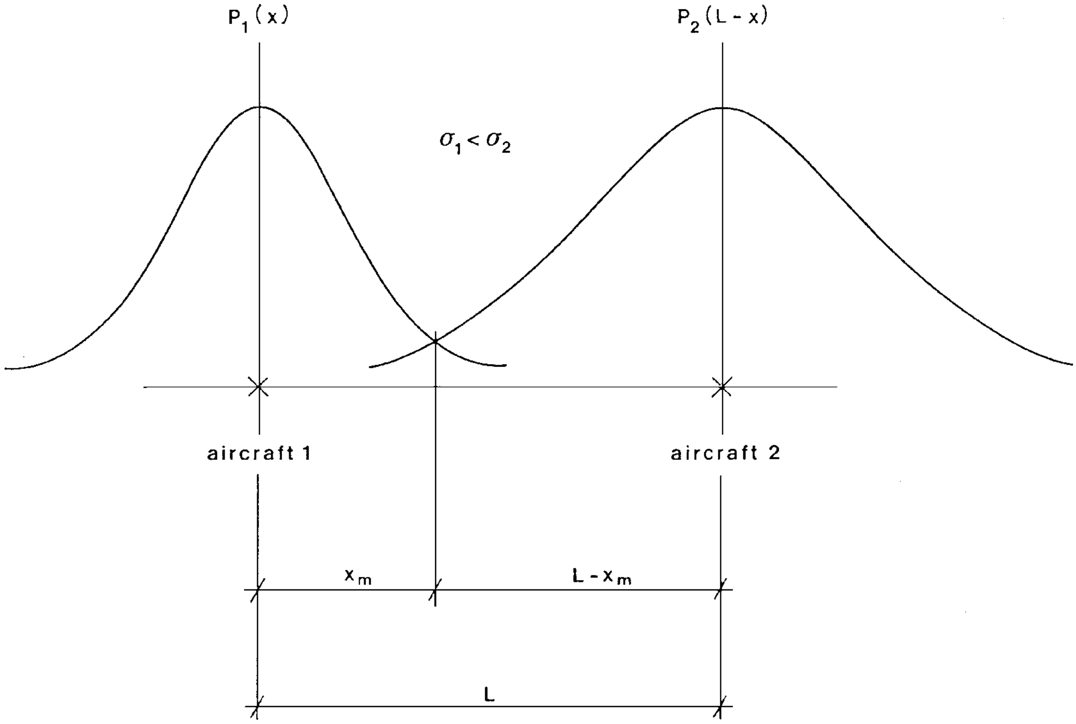

Fig. 4 If aircraft have dissimilar rms position errors $\sigma_2 > \sigma_1$, position x_m of most likely collision is closer to the intended position of the aircraft 1, with a smaller position error.

In the case of similar aircraft ($\sigma_1 = \sigma_2 \equiv \sigma$), the maximum probability of collision simplifies to

$$P_m = [1/(2\pi\sigma^2)]\exp\{-[L/(2\sigma)]^2\} \tag{12}$$

where σ^2 replace $\sigma_1\sigma_2$ in the factor of the exponential and $[(\sigma_1)^2 + (\sigma_2)^2]/2$ in the argument of the exponential.

C. Cumulative Probability of Collision

The result (11) is the maximum probability of collision, which occurs at the most likely position $x = x_m$ for a collision (7). Collisions can also occur at all other lateral positions $-\infty < x < +\infty$, with a smaller probability as x deviates more from x_m. To include all possibilities for lateral (other possibilities for collision will be considered in future work) collision, a cumulative probability of collision is defined by

$$\bar{P} \equiv \int_{-\infty}^{+\infty} P_{12}(x)\,\mathrm{d}x \tag{13}$$

Substitution of Eq. (4) leads to

$$\bar{P} = \left[\frac{1}{2\pi\sigma_1\sigma_2}\right]\exp\left\{-\frac{L^2/2}{(\sigma_2)^2}\right\}\int_{-\infty}^{+\infty}\exp\left\{-\left(\frac{x^2}{2}\right)\left[(\sigma_1)^{-2} + (\sigma_2)^{-2}\right] + xL(\sigma_2)^{-2}\right\}\mathrm{d}x \tag{14}$$

where the integral can be reduced to a Gaussian integral:

$$\int_{-\infty}^{+\infty}\exp(-y^2)\,\mathrm{d}y = \sqrt{\pi} \tag{15}$$

To do this, the argument of the exponential in braces in Eq. (14) is written as the sum of the square of a linear function of x, plus a constant term independent of x:

$$-(x^2/2)\left[(\sigma_1)^{-2} + (\sigma_2)^{-2}\right] + xL(\sigma_2)^{-2} = -\Big[\big(x/\sqrt{2}\big)\times\sqrt{(\sigma_1)^{-2} + (\sigma_2)^{-2}} - \big(L/\sqrt{2}\big)(\sigma_2)^{-2}\Big/\sqrt{(\sigma_1)^{-2} + (\sigma_2)^{-2}}\Big]^2 + \left[L^2/2(\sigma_2)^4\right]\Big/\left[(\sigma_1)^{-2} + (\sigma_2)^{-2}\right] \tag{16}$$

suggesting the change of variable

$$y \equiv \big(x/\sqrt{2}\big)\sqrt{(\sigma_1)^{-2} + (\sigma_2)^{-2}} - \big(L/\sqrt{2}\big)(\sigma_2)^{-2}\Big/\sqrt{(\sigma_1)^{-2} + (\sigma_2)^{-2}} \tag{17}$$

which can be made,

$$\mathrm{d}x = \sqrt{2}\left[(\sigma_1)^{-2} + (\sigma_2)^{-2}\right]^{-\frac{1}{2}}\mathrm{d}y \tag{18}$$

in the integral (14), namely,

$$\bar{P} = \left[\frac{1}{\sqrt{2}\pi\sigma_1\sigma_2}\right]\left[(\sigma_1)^{-2} + (\sigma_2)^{-2}\right]^{-\frac{1}{2}}\exp\left\{-\left[\frac{L^2}{2(\sigma_2)^2}\right]\times\left[1 - \frac{(\sigma_2)^{-2}}{(\sigma_1)^{-2} + (\sigma_2)^{-2}}\right]\right\}\int_{-\infty}^{+\infty}\exp(-y^2)\,\mathrm{d}y \tag{19}$$

Using Eq. (15) simplifies the cumulative probability of collision to

$$\bar{P} = \left\{2\pi\left[(\sigma_1)^2 + (\sigma_2)^2\right]\right\}^{-\frac{1}{2}}\exp\left\{-(L^2/2)\Big/\left[(\sigma_1)^2 + (\sigma_2)^2\right]\right\} \tag{20}$$

This is similar to the Gaussian probability distribution for the position error (2), substituting the variance $(\sigma_1)^2$ for one aircraft by the total variance $(\sigma_1)^2 + (\sigma_2)^2$ for two aircraft. It applies to dissimilar aircraft, and for similar aircraft ($\sigma_1 = \sigma_2 \equiv \sigma$) simplifies to

$$\bar{P} = \left[1\Big/\big(2\sigma\sqrt{\pi}\big)\right]\exp\{-[L/(2\sigma)]^2\} \tag{21}$$

where $2\sigma^2$ replace $(\sigma_1)^2 + (\sigma_2)^2$.

III. Reduction of the Case of Dissimilar Aircraft to Quasi Similarity

The probabilities of collision for dissimilar aircraft are brought to a form as close as possible to those for similar aircraft, by introducing the arithmetic (Sec. III.A) and geometric (Sec. III.B) means of the variances of position errors. In this way it is possible to have a complete similarity for the cumulative probability of collision, but not for the maximum probability of collision. The latter requires the introduction of a dissimilarity factor and a dissimilarity function (Sec. III.C), whose effects are assessed.

A. Arithmetic Mean of Variances of Position Errors

Consider two dissimilar aircraft in the sense that the rms position errors σ_1 and σ_2 may be different, and thus also the variances $(\sigma_1)^2$ and $(\sigma_2)^2$. The arithmetic mean of the variances is defined by

$$2\bar{\sigma}^2 \equiv (\sigma_1)^2 + (\sigma_2)^2 \tag{22}$$

It is clear that the rms position errors σ_1 and σ_2 appear in the cumulative probability of collision (20) through the arithmetic mean $\bar{\sigma}$ in the variances:

$$\bar{P} = \left[1\big/\left(2\bar{\sigma}\sqrt{\pi}\right)\right]\exp\{-[L/(2\bar{\sigma})]^2\} \tag{23}$$

This is the same formula as for identical aircraft ($\sigma_1 = \sigma_2 \equiv \sigma = \bar{\sigma}$):

$$\bar{P} = \left[1\big/\left(2\sigma\sqrt{\pi}\right)\right]\exp\{-[L/(2\sigma)]^2\} \tag{24}$$

replacing σ by $\bar{\sigma}$. Thus, the cumulative probability of collision for aircraft with dissimilar variances $(\sigma_1)^2$ and $(\sigma_2)^2$ for the position error, is the same [Eq. (20)] as for identical aircraft [Eq. (23)] whose variance $(\bar{\sigma})^2$ for the position error is the arithmetic mean of variances (22).

B. Geometric Mean of Variances and Dissimilarity Function

The complete reduction of the case of dissimilar aircraft to that of similar aircraft is possible for the cumulative probability of collision (23), which involves Eq. (20) only the arithmetic mean of the variances (22), but not the maximum probability of collision (11), that is,

$$P_m = [1/(2\pi\sigma_1\sigma_2)]\exp\{-[L/(2\bar{\sigma})]^2\} \tag{25}$$

which involves also the geometric mean of the variances,

$$\sigma_1\sigma_2 = \left[(\sigma_1)^2(\sigma_2)^2\right]^{\frac{1}{2}} \tag{26}$$

The maximum probability of collision (25) can be put in a form resembling that of Eq. (12) for similar aircraft:

$$P_m = f[1/(2\pi\bar{\sigma}^2)]\exp\{-[L/(2\bar{\sigma})]^2\} \tag{27}$$

only by introducing a dissimilarity function

$$f \equiv \bar{\sigma}^2\big/(\sigma_1\sigma_2) = (\sigma_1/\sigma_2 + \sigma_2/\sigma_1)/2 \tag{28}$$

which is the ratio of the arithmetic (22) and geometric (26) means of the variances.

C. Ratio of RMS Position Errors and Dissimilarity Factor

The dissimilarity function

$$f(\lambda) \equiv (\lambda + 1/\lambda)/2 \tag{29a}$$

$$\lambda \equiv \sigma_2/\sigma_1 \tag{29b}$$

depends only on the ratio of the rms position errors, which may be called the dissimilarity factor λ. The dissimilarity function is unchanged if the two aircraft are interchanged:

$$f(\lambda) = f(1/\lambda) \tag{30}$$

From its first two derivatives:

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{[1-(1/\lambda^2)]}{2} \tag{31a}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} = \frac{1}{\lambda^3} \tag{31b}$$

it follows that the dissimilarity function has a minimum of unity for similar aircraft:

$$\sigma_2 = \sigma_1 : \lambda = 1 \tag{32a}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = 0 \tag{32b}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} > 0 \tag{32c}$$

and for dissimilar aircraft it is always greater than unity:

$$f(\lambda) \geq f_{\min} = f(1) = 1 \tag{33}$$

For very dissimilar aircraft f scales as $\lambda/2$ for large λ and as $1/(2\lambda)$ for small λ:

$$f(\lambda) \sim \begin{cases} \lambda/2 & \text{if} \quad \lambda = \sigma_2/\sigma_1 \gg 1 \quad (34a) \\ 1/(2\lambda) & \text{if} \quad \lambda = \sigma_2/\sigma_1 \ll 1 \quad (34b) \end{cases}$$

as can be seen in Fig. 5. Because the dissimilarity function is algebraic and multiplies an exponential in the maximum probability of collision, its effect in decreasing the latter is less significant than the effect of changing the mean rms position error $\bar{\sigma}$, as will be shown next.

IV. Application to Aircraft Separation in Flight or on the Ground

The cumulative (23) and maximum (27) probabilities of collision can be applied to similar or dissimilar aircraft and are calculated as a function of rms position error, for several values of the minimum separation distance: the 5 n mile for horizontal separation in controlled airspace, which could also apply to longitudinal separation in a landing sequence (Sec. IV.A) (Table 1); 1000 ft for vertical separation in flight, which could also apply to the longitudinal separation on taxiing on a runway (Sec. IV.B); and 60 n mile for horizontal separation in transoceanic airspace (Sec. IV.C). Note that taking into account the finite size of the two aircraft, L_1 and L_2, implies reducing the minimum separation distance from L to $L - L_1 - L_2$. This correction becomes irrelevant if the separation distance is much larger than the size of the aircraft, $L \gg L_1, L_2$, for example, for the 60-n mile separation in transoceanic airspace.

A. 5-Nautical Mile Horizontal Separation En Route or Longitudinal Separation on Landing

Formula (23) for the cumulative probability of collision per nautical mile flown,

$$\bar{P} = (0.282095/\bar{\sigma})\exp[-L^2/(4\bar{\sigma}^2)] \tag{35}$$

can be applied for any minimum separation distance, for example, $L_h = 5$ n mile lateral separation in controlled air space,

$$\bar{P}_h = (0.282095/\bar{\sigma})\exp(-6.25/\bar{\sigma}^2) \tag{36}$$

Table 1 Dissimilarity function vs dissimilarity parameter: $f(\lambda) \equiv (\lambda + 1/\lambda)/2 = f(1/\lambda)$

λ	f	$1/\lambda$	λ	f	$1/\lambda$
1	1	1	11	5.545(45)	0.09(09)
2	1.25	0.5	12	6.0416(6)	0.083(3)
3	1.6(6)	0.3(3)	13	6.53846	0.076931
4	2.125	0.25	14	7.03571	0.071429
5	2.6	0.2	15	7.53(3)	0.06(6)
6	3.083(3)	0.16(6)	16	8.03125	0.0625
7	3.57143	0.14286	17	8.52941	0.588235
8	4.0625	0.125	18	9.027(7)	0.05(5)
9	4.5(5)	0.1(1)	19	9.52632	0.05263
10	5.05	0.1	20	10.025	0.05

Table 2 Equivalent safety metrics for 5 n mile lateral separation in controlled airspace, or 5-mile longitudinal separation in a landing sequence

Arithmetic mean of variances $\bar{\sigma}_h$, n mile	Cummulative probability of collision $\bar{P}_h$, n mile	Maximum probability of collision P_{mh} (per square nautical mile)		
		$\lambda = 1$	$\lambda = 3$	$\lambda = 10$
3	4.70×10^{-3}	8.83×10^{-3}	1.47×10^{-2}	4.46×10^{-2}
2	3.01×10^{-4}	8.34×10^{-3}	1.39×10^{-2}	4.21×10^{-2}
1	5.45×10^{-4}	3.07×10^{-4}	5.11×10^{-4}	1.55×10^{-3}
0.9	1.40×10^{-4}	8.76×10^{-5}	1.46×10^{-4}	4.42×10^{-4}
0.8	2.03×10^{-5}	1.43×10^{-5}	2.38×10^{-5}	7.22×10^{-5}
0.7	1.16×10^{-6}	9.38×10^{-7}	1.56×10^{-6}	4.74×10^{-6}
0.6	1.36×10^{-8}	1.28×10^{-8}	2.13×10^{-8}	6.46×10^{-8}
0.5	7.83×10^{-12}	8.84×10^{-12}	1.47×10^{-11}	4.46×10^{-11}
0.44	3.84×10^{-15}	7.84×10^{-15}	1.31×10^{-14}	3.96×10^{-14}
0.4	7.65×10^{-18}	1.08×10^{-17}	1.80×10^{-17}	5.45×10^{-17}
0.35	5.06×10^{-23}	9.03×10^{-23}	1.51×10^{-22}	4.56×10^{-22}
0.3	6.51×10^{-31}	1.23×10^{-30}	2.05×10^{-30}	6.21×10^{-30}
0.25	4.20×10^{-44}	9.47×10^{-44}	1.58×10^{-43}	4.78×10^{-43}

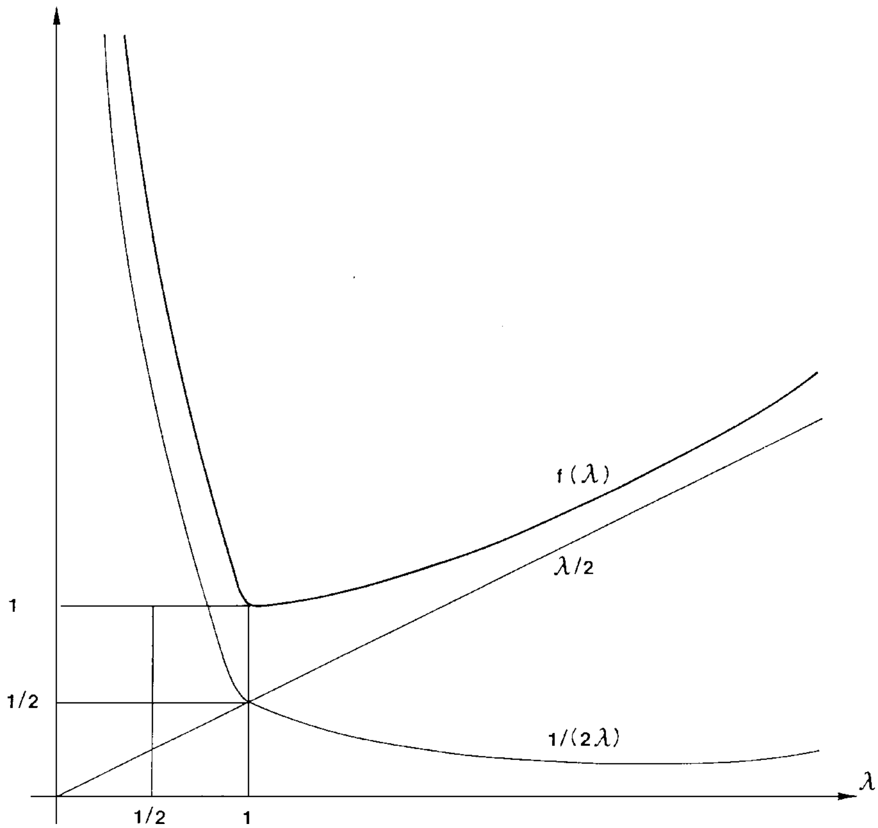

Fig. 5 Dissimilarity function f has minimum of unity for similar aircraft $\lambda = \sigma_2/\sigma_1 = 1$ and increases $\infty > f > 1$ as λ differs from unity $0 \le \lambda < 1$ or $1 < \lambda < \infty$. Note $f = \infty$ or $\lambda = 0, \infty$ are excluded because $\sigma_1 = 0$ (or $\sigma_2 = 0$) lead to vanishing $P_1 = 0$ (or $P_2 = 0$) Gaussian probability distributions.

Its values are given in Table 2 as a function of the arithmetic mean $\bar{\sigma}$ of variances $\bar{\sigma}^2$ of position errors (22). The maximum probability of collision per square nautical mile flown [Eq. (23)]

$$P_m = (0.159155/\bar{\sigma}^2)\exp[-L^2/(4\bar{\sigma}^2)] \cdot f \quad (37)$$

for example, for the lateral separation in controlled airpace,

$$P_{mh} = (0.159155/\bar{\sigma}^2)\exp(-6.25/\bar{\sigma}^2)f \quad (38)$$

depends not only on $\bar{\sigma}$, but also on the dissimilarity function (29a), which is indicated in Table 1 for several values of the dissimilarity parameter (29b). In Table 2, the maximum probability of collision is indicated for identical aircraft $\lambda = 1$, for rms position errors in the ratio 3, $\lambda = 3$ or $\lambda = \frac{1}{3}$, or 10, $\lambda = 0.1$ or $\lambda = 10$. The results would be similar for a 5-n mile longitudinal separation in a landing sequence.

It is seen from Table 2 that an arithmetic mean variance $\bar{\sigma}^2$ corresponding to a rms position error $\bar{\sigma} = 0.5$ n mile leads to a very low cumulative probability of collision $\bar{P}_h \le 7.83 \times 10^{-12}$/n mile. For the Concorde flying at Mach 2, that is, $V = 1146$ kn, the probability of collision per flight hour would be $\bar{P}_h V \le 8.97 \times 10^{-9}$/h, that is, close to the ICAO TLS standard of $S \le 5 \times 10^{-9}$/h. For a half-great circle tour of the Earth, $D = 2 \times 10^4$ km $= 1.08 \times 10^4$ n mile, the probability of collision would still be small, $\bar{P}_h D \le 8.45 \times 10^{-8}$. The maximum probability of collision P_{mh} applies per square nautical mile, that is, nautical mile flown by each aircraft and is, thus, more restrictive. For a rms position error, $\bar{\sigma} = 0.44$ n mile, the maximum probability of collision is $P_{mh} \le 7.84 \times 10^{-15}$/n mile^2 for identical aircraft. The modified ICAO standard $R = 5 \times 10^{-9}$/h^2, would apply to two aircraft with geometric mean of velocities $V_{mh} = \sqrt{(R/P_{mh})} \le 799$ kn, for identical aircraft. For aircraft with rms position errors in the ratio $\lambda = 3$, then $P_{mh} \le 1.31 \times 10^{-14}$/n mile^2, and the modified ICAO standard is met for velocities up to $V_{mh} \le 618$ kn. For aircraft with rms position errors in the ratio 10, $\lambda = 10$, then $P_{mh} \le 3.96 \times 10^{-14}$ and $V_{mh} \le 355$ kn, which is a bit too low. Thus, a rms position error of $\bar{\sigma} \le 0.4$ mile and a ratio of rms position errors $\lambda \le 10$ guarantee the original and modified ICAO standard of safety, for lateral separation in controlled airspace.

B. 1000-Feet Vertical Separation in Flight or Horizontal Separation on Ground

As an example of application to another separation distance, in controlled airspace, instead of the lateral separation of $L_h = 5$ n mile, the vertical separation could be considered with 1000 ft, $L_V = 1000$ ft $= 0.1645$ n mile. The same results would apply for the same horizontal spacing on ground, for example, a takeoff sequence at an airport with a longitudinal separation of 1000 ft. The cumulative probability of collision is given [Eq. (35)] in this case by

$$\bar{P}_v = (0.282095/\bar{\sigma}_v)\exp\left(-6.7642 \times 10^{-3}\big/\bar{\sigma}_v^2\right) \qquad (39)$$

in terms of the rms position error $\bar{\sigma}$, calculated from the variance $\bar{\sigma}$ as an arithmetic mean of the variances of two aircraft (22). The maximum probability of collision (37), namely,

$$P_{mv} = \left(0.159155\big/\bar{\sigma}_v^2\right)\cdot f \cdot \exp\left(-6.7642 \times 10^{-3}\big/\bar{\sigma}_v^2\right) \qquad (40)$$

depends also on the dissimilarity function f [Eq. (29a)] given in Table 1 for several values of the dissimilarity parameter (29b) or the ratio of the rms position errors of the two aircraft. Table 3 indicates the cumulative, P_v, and maximum, P_{mv}, probabilities of collision, for several values of the rms position error $\bar{\sigma}_v$ and three values of the dissimilarity parameter.

In the preceding case of 5-mile separation, $L_h = 5$ n mile the ICAO safety standard was met by a rms position error $\sigma_h = 0.4$ n mile in the ratio $L_h/\sigma_h = 12$, for a dissimilarity factor up to 10, $\sigma_2/\sigma_1 \le 10$ (taking, by convention, $\sigma_2 \ge \sigma_1$). This, in the case of vertical separation $L_V/\sigma_v = 12$, suggests an rms altitude error $\sigma_v = 80$ ft, for which the cummulative probability of collision $\bar{P}_v \le 3.44 \times 10^{-14}$/n mile is small even for a half-grand circle trip on the Earth, $\bar{P}_v D = 3.71 \times 10^{-10}$; the ICAO criterion $S \le 5 \times 10^{-9}$ per flight hour, would be met handsomely by the fastest commercial aircraft flying, the Concorde, $V \le 1146$ kn, because $\bar{P}_v V = 3.94 \times 10^{-11} \ll R$. The maximum probability of collision $P_{mv} \le 7.30 \times 10^{-18}$/n mile^2 for aircraft with identical rms altitude errors $\sigma_v = 80$ ft, the modified ICAO criterion $R \le 5 \times 10^{-9}$/h^2 would be met by two aircraft with geometric mean of velocities $V_{mv} \le \sqrt{(R/P_{mv})} = 2.61 \times 10^4$ kn. If the aircraft had dissimilar rms altitude errors in the ratio 3, $\lambda = 3$, then $P_{mv} \le 1.21 \times 10^{-17}$ for velocities up to $V_{mv} \le 2.03 \times 10^4$ kn, and for a dissimilarity factor of 10, $\lambda = 10$, then $P_{mv} \le 3.69 \times 10^{-17}$ and $V_{mv} \le 1.16 \times 10^4$ kn, which includes all commercial airliners.

C. 60 Nautical Mile Horizontal Separation in Transoceanic Airspace

The preceding applications have concerned relatively small separations in flight or on the ground, respectively, for en route flying and landing or takeoff sequences. For an application with a larger separation of 60 n mile $L_t = 60$ n mile, the case of lateral separation in transoceanic flight is considered. The cumulative probability of collision is given [Eq. (35)] by

$$\bar{P}_t = (0.282095/\bar{\sigma}_t)\exp\left(-900\big/\bar{\sigma}_t^2\right) \qquad (41)$$

per nautical mile, as a function of the rms position error, calculated from the variance $\bar{\sigma}^2$ as an arithmetic mean of variances. The maximum probability of collision (37) per square nautical mile,

$$P_{mt} = \left(0.159155\big/\bar{\sigma}_t^2\right)\cdot f \cdot \exp\left(-900\big/\bar{\sigma}_t^2\right) \qquad (42)$$

Table 3 Equivalent safety metrics for vertical separation of 1000 ft in flight or longitudinal separation of 1000 ft for aircraft in takeoff sequence on the ground

RMS altitude error $\bar{\sigma}_v$, ft	Cummulative probability of collision $\bar{P}_v$/n mile	Maximum probability of collision P_{mv}/n mile^2		
		$\lambda = 1$[a]	$\lambda = 3$[a]	$\lambda = 10$[a]
500	9.55×10^{-1}	6.33×10^{-3}	1.06×10^{-2}	3.20×10^{-2}
400	4.91×10^{-1}	5.64×10^{-3}	9.42×10^{-2}	2.85×10^{-2}
300	2.67×10^{-1}	2.97×10^{-3}	3.87×10^{-3}	1.17×10^{-2}
200	1.25×10^{-2}	2.08×10^{-4}	3.47×10^{-4}	1.05×10^{-3}
100	1.80×10^{-10}	5.98×10^{-12}	9.95×10^{-12}	3.01×50^{-11}
90	5.66×10^{-13}	2.10×10^{-14}	3.49×10^{-14}	1.06×10^{-14}
80	3.44×10^{-14}	7.30×10^{-18}	1.21×10^{-17}	3.69×10^{-17}
70	1.29×10^{-21}	6.11×10^{-23}	1.01×10^{-22}	3.08×10^{-22}
60	1.49×10^{-29}	8.30×10^{-31}	1.34×10^{-30}	4.19×10^{-30}
50	9.63×10^{-43}	6.41×10^{-44}	1.07×10^{-43}	3.23×10^{-43}

[a]Dissimilarity factor $\lambda = \sigma_2/\sigma_1$.

Table 4 Equivalent safety metrics for transoceanic horizontal separation $L_t = 60$ n mile

RMS position error σ_t, n mile	Cummulative probability of collision $\bar{P}_t$/n mile	Maximum probability of collision P_{mt}/n mile^2		
		$\lambda = 1$[a]	$\lambda = 3$[a]	$\lambda = 10$[a]
30	2.61×10^{-3}	3.70×10^{-5}	7.28×10^{-4}	1.87×10^{-4}
20	1.12×10^{-3}	2.38×10^{-5}	3.97×10^{-5}	1.20×10^{-4}
10	2.62×10^{-6}	1.12×10^{-7}	1.87×10^{-7}	5.66×10^{-7}
9	3.53×10^{-7}	1.67×10^{-8}	2.78×10^{-6}	8.43×10^{-6}
8	1.39×10^{-7}	7.39×10^{-9}	1.23×10^{-8}	3.73×10^{-8}
7	3.20×10^{-7}	1.95×10^{-11}	3.25×10^{-11}	9.85×10^{-11}
6	4.92×10^{-13}	3.49×10^{-14}	5.82×10^{-14}	1.76×10^{-13}
5.5	7.08×10^{-15}	6.31×10^{-16}	1.05×10^{-15}	3.19×10^{-15}
5	9.90×10^{-18}	8.37×10^{-19}	1.40×10^{-18}	4.23×10^{-18}
4.5	2.36×10^{-21}	2.23×10^{-22}	3.72×10^{-22}	1.13×10^{-21}
4.0	1.98×10^{-26}	2.10×10^{-27}	3.50×10^{-27}	1.06×10^{-26}
3.5	7.52×10^{-34}	9.05×10^{-35}	1.51×10^{-34}	4.57×10^{-34}
3	2.64×10^{-45}	3.74×10^{-46}	6.23×10^{-46}	1.89×10^{-45}

[a]Dissimilarity factor.

depends also on the dissimilarity function (29a) given in terms of the dissimilarity factor (29b) in Table 1.

From the results in Table 4, the case rms position error $\bar{\sigma}_t = 6$ n mile $\frac{1}{10}$ of the separation distance $L_t = 60$ n mile is highlighted. The cumulative probability of collision, $\bar{P}_t \leq 4.92 \times 10^{-13}$/mile, corresponds to $\bar{P}_t D = 5.31 \times 10^{-9}$ for a half-great-circle tour of the Earth. For the cruise speed of the Concorde, $V = 1146$ kn, it leads to a probability of collision per flying hour $\bar{P}_t V = 5.64 \times 10^{-10}$ below the ICAO TLS limit $S = 5 \times 10^{-9}$/h. For a slightly smaller rms position error, $\sigma_t = 5.5$ n mile the maximum probability of collision $P_{mt} \leq 6.31 \times 10^{-16}$/n mile^2 leads to a probability of collision $P_{mt} D^2 = 7.35 \times 10^{-8}$ for a half-great-circle-trip of the Earth, in the case of similar aircraft. In the same case of similar aircraft, the maximum velocity that meets the modified ICAO requirement is $V_{mt} = \sqrt{(R/P_{mt})} \leq 2.81 \times 10^3$ kn. For aircraft with rms position errors in the ratio $\lambda = 3$, then $P_{mt} \leq 1.05 \times 10^{-15}$ leads to $V_{mt} \leq 2.18 \times 10^3$ kn, and for $\lambda = 10$, then $P_{mt} \leq 3.19 \times 10^{-15}$ and $V_{mt} \leq 1.25 \times 10^3$ kn. Thus, the modified ICAO standard is met for a rms position error $\sigma_t = 5.5$ n mile and aircraft dissimilarity factor up to $\lambda = 10$. In fact, $P_{mt} \leq R/V^2 = 3.81 \times 10^{-15}$/mile^2 for Concorde $V = 1146$ kn implying a dissimilarity function $f = P_{mt}/P_{mtc} = 6.04 = \lambda$, which corresponds to a large dissimilarity factor $\lambda \leq 12$.

V. Discussion and Summary

It has been shown that the ICAO standard of probability of collision 5×10^{-9} per flight hour is met, if the aircraft trajectories respect a minimum separation distance L and the rms position error does not exceed $\bar{\sigma} \leq L/12$. The value of $\bar{\sigma}$ is the rms position error for similar aircraft and is given by Eq. (22) for aircraft with dissimilar rms position errors. The result holds for the cumulative probability of collision summed over all possible collision positions. It also holds for the maximum collision probability, at the position of most likely collision using the modified ICAO standard of 5×10^{-9} per flight hour squared. In the latter case, it holds for similar or dissimilar aircraft, with ratios of rms position errors up to 10, $\sigma_2/\sigma_1 \leq 10$. These results can be applied to any separation, in flight or on the ground, for example, the lateral separation for en route flying in controlled (transoceanic) airspace $L_h = 5$ n mile ($L_t = 60$ n mile) leads to horizontal rms position error $\sigma_r \leq 0.45$ n mile ($\sigma_t = 5.5$ n mile); the vertical separation of $L_v = 1000$ ft leads to an rms altitude error $\sigma_v \leq 80$ ft; the same results in Tables 1–3 could be applied to the longitudinal separation for aircraft in a landing or takeoff sequence as well as to aircraft and ground vehicles in an airport.

Tables 1–3 allow the use of the rms position error as a safety test. The rms position error of the actual trajectory relative to the intended trajectory can be measured or monitored, from real flight data and fast time simulations. If the intended trajectories never violate the minimum separation distance, Tables 1–3 give an upper bound for the probability of collision. If this upper bound is less than the desired collision probability, the situation is assuredly safe (Fig. 6). If the upper bound is higher than the desired collision probability, then the situation is potentially unsafe; if the actual probability of collision, which is lower than the upper bound, is still above the desired probability of collision, the situation is actually unsafe. In a potentially unsafe situation, the rms position error exceeds the safe value. The analysis of how this exceedance occurs is a diagnostic for the cause of the potentially unsafe situation, for example, it could be due to 1) excessive inaccuracy of the navigation system, 2) pilot error, 3) controller error, 4) atmospheric disturbances causing excessive drift between position fixes, 5) low visibility allowing position errors to go undetected, 6) excessive pilot or controller workload degrading position monitoring, etc. Other, more detailed and cumbersome metrics, could aid in diagnosing causes of loss of safety. In this case, the rms position error would be a quick and simple test to detect potentially unsafe cases. The diagnostic of the causes for unsafety would be made by assessing the various contributions to the rms position error or using other safety metrics.

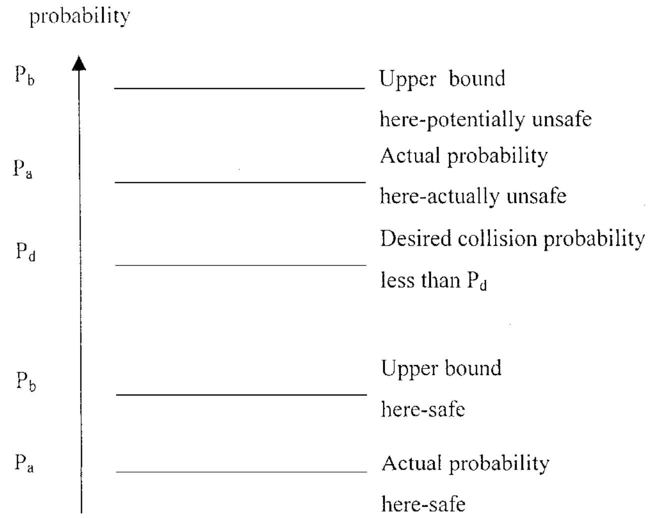

Fig. 6 Relation between desired P_d and actual P_a probability of collision and upper bound P_b for probability of collision $P_b \leq P_a$, for several conditions: safe $P_b \leq P_d$, potentially unsafe $P_d < P_b$ and actually safe $P_a < P_d$.

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